On Riccati equations describing impedance relations for forward and backward excitation in the one-dimensional cochlea model

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(Received 23 September 1985; accepted for publication 16 October 1986)

Recent experimental observations of otoacoustic emissions suggest the existence of spontaneous emitters of sound on the basilar membrane. These tend to send off waves not only in the normal direction of propagation. It is therefore significant to study the environmental conditions such an emitter finds inside the cochlea. The impedance relations seen by these emitters are described by the Riccati equation for an inhomogeneous transmission line. The results reported in this paper differ considerably for forward and backward excitation. This reflects the quite different behavior of the cochlea pertaining to waves traveling forward and backward. Because of reflections, backward waves cannot be treated with the Liouville–Green approximation.

PACS numbers: 43.63.Bq

INTRODUCTION

Recent experimental observations of otoacoustic emissions suggest the existence of spontaneous emitters of sound on the basilar membrane (cf. Kemp, 1978; de Wit, 1985). These require the consideration of several new physical properties which are not taken into account by classical cochlea models. One aspect is the stability of an emitter, which is investigated in another paper (de Boer *et al.*, 1985). In this paper we address the problem of the propagation of waves on the basilar membrane induced by such an emitter. The emitter tends to send off waves not only in the normal direction of propagation towards the *helicotrema* (forward direction), but also backward towards the *stapes*. It is therefore necessary to study the behavior of such backward traveling waves.

In order to concentrate on the basic physical properties of the cochlea we use the simple and well-studied one-dimensional long-wave model (Zwislocki, 1953; de Boer, 1980). Making use of the inhomogeneous transmission line analog (cf. de Boer, 1984), this approach allows us to calculate the impedances inside the cochlea as seen by an emitter. It has been found that a transmission line of the type of the cochlea treats waves quite differently, depending upon their direction (de Boer and Viergever, 1982). It is the purpose of this paper to discuss this effect in some more detail.

I. THE RICCATI EQUATIONS

In a homogeneous transmission line, propagation of energy by way of waves is controlled by the line's characteristic impedance. For an inhomogeneous line the concept of characteristic impedance, even when interpreted locally, is in general less useful. It is more convenient to consider energy propagation in terms of two types of "local impedance." Each of the impedances is the solution of a Riccati equation (cf. Meinke and Gundlach, 1968; de Boer *et al.*, 1985).

In the following, we derive the two versions of the Riccati equation adequate to treat forward and backward waves. The numerical solution for a particular set of parameters is presented, and then we discuss the physical implications of the results.

Let x denote the spatial coordinate of a transmission line, increasing to the right. Let ω be the angular frequency of the excitation. Define the local impedance $\mathscr{R}(x,\omega)$ as the impedance of the right part of the transmission line when it is cut at location x. To measure it, we would apply a source to the terminal pair of the right part. Correspondingly, the local impedance $\mathscr{L}(x,\omega)$ is the impedance of the left part of the transmission line, when cut at x.

Consider now an infinitesimal element of length dx of the line at location x as shown in Fig. 1. Let $Z(x,\omega)$ be the series impedance density and $Y(x,\omega)$ the shunt admittance density at location x for an excitation with angular frequency ω. Following the definition of $\mathscr{R}$ we know that the element is loaded at its right-hand terminal pair by $\mathscr{R}(x+dx,\omega)$, i.e., the impedance of the part of the cochlea to the right of the location $x+dx$. Using Kirchhoff's laws we can evaluate the impedance presented at the element's left-hand terminal pair. Then, keeping in mind that this by definition is equal to $\mathscr{R}(x,\omega)$ and neglecting second-order terms in dx give rise to the Riccati equation for $\mathscr{R}$:

$$\mathscr{R}'(x,\omega) = +Y(x,\omega)\mathscr{R}^2(x,\omega) - Z(x,\omega). \qquad (1a)$$

The prime denotes differentiation with respect to x.

We can also look at the same situation the other way

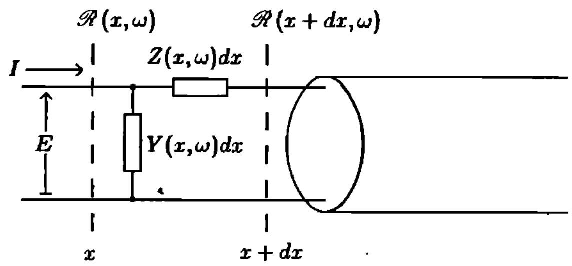

FIG. 1. Infinitesimal element of length dx of the transmission line. Here, E represents voltage, I stands for current, x denotes the spatial coordinate of the transmission line, $Z(x,\omega)$ is the series impedance density, $Y(x,\omega)$ is the shunt admittance density, and $\mathscr{R}(x,\omega)$ is defined as the impedance of the right part of the transmission line when cut at x.

round. The infinitesimal element is loaded at the left by the impedance $\mathscr{L}(x,\omega)$. It then should present the impedance $\mathscr{L}(x+dx,\omega)$ at its right-hand terminals. In this way we obtain the Riccati equation for $\mathscr{L}$:

$$\mathscr{L}'(x,\omega) = -Y(x,\omega)\mathscr{L}^2(x,\omega) + Z(x,\omega). \qquad (1b)$$

Note the difference in signs of Eqs. (1a) and (1b).

Let us consider the situation where we start with some load impedance at the right end of some interval of the transmission line. We want to determine the impedance of the "complete" system at the left end, i.e., the load impedance as it is transformed by the transmission line. We should integrate Eq. (1a) over this interval from right to left. The situation described above is relevant in the case where the cochlea is driven from the left, and it yields the impedance loading the source. If, instead, the transmission line is driven from the right, we should integrate Eq. (1b), starting with the load impedance at the left end and yielding the impedance loading the source at the right end.

II. THE COCHLEA MODEL

We now apply this to the case of the one-dimensional cochlea model. This model is analogous to the transmission line situation (cf. de Boer, 1984). Let us represent the pressure across the basilar membrane by the voltage and the particle velocity of the fluid by the current density.[1] We then yield the series impedance density $Z(x,\omega)$ to be analogous to the specific longitudinal impedance of the cochlea fluids:

$$Z(x,\omega) = Z(\omega) = i\omega\rho, \qquad (2)$$

where ρ is the fluid density. The shunt admittance density can be represented by the basilar membrane impedance $\zeta(x,\omega)$ as follows:

$$Y(x,\omega) = 2/h\zeta(x,\omega),$$

with (3)

$$\zeta(x,\omega) = i\omega M(x) + R(x) + S(x)/i\omega.$$

Here, h is the effective height of the cochlea (i.e., the cross section of one scala divided by the width of the basilar membrane), $M(x)$ is the mass density, $R(x)$ is the resistance density, and $S(x)$ is the stiffness density at location x. In the one-dimensional long-wave model, the x dependence of the parameters is assumed to be as follows:

$$M(x) = M_0, \quad S(x) = S_0 e^{-\alpha x},$$

and (4)

$$R(x) = \delta[S(x)M(x)]^{1/2},$$

where α is the space constant, δ is the damping fraction, and the subscript zero refers to the *stapes* location at $x = 0$. In order to fit experimental data, a particular set of parameters is chosen following de Boer (1980); see Table I. It should be noted that M_0 and S_0 are positive. This implies that there is a point x_r where the imaginary part of $Y(x,\omega)$ vanishes and a true resonance occurs.

TABLE I. Values of the parameters of the one-dimensional cochlea model.

Parameter	Value	Unit
M_0	5×10^{-4}	g mm^{-2}
S_0	10^7	μN mm^{-3}
α	0.3	mm^{-1}
δ	0.05	1
h	1.0	mm
ρ	10^{-3}	g mm^{-3}

With the series impedance density and the shunt admittance density given as in Eqs. (2)–(4), Eq. (1a) can be written in a form with only two independent parameters:

$$\frac{d\tilde{\mathscr{R}}(\xi)}{d\xi} = \frac{\tilde{\mathscr{R}}^2(\xi)}{i(1-e^{-\xi}) + \delta e^{-\xi/2}} - i\tilde{\rho}, \qquad (5)$$

with

$$\xi = \alpha x + \ln(M_0\omega^2/S_0),$$

$$\tilde{\mathscr{R}}(\xi) = 2\mathscr{R}(x,\omega)/h\omega M_0\alpha,$$

and

$$\tilde{\rho} = 2\rho/hM_0\alpha^2.$$

The parameter δ remains a damping constant. Its value is the same as before (see Table I). The behavior of the solution of Eq. (5) will not change strongly if δ remains positive and is not so large as to inhibit the basilar membrane from oscillating. The parameter $\tilde{\rho}$ has the meaning of an inverse coupling constant. With the numbers of Table I, $\tilde{\rho}$ is computed as $44.\bar{4}$. This value is thus derived from a realistic parametrization of the cochlea. The above transformation can also be applied to Eq. (1b).

In this paper we wish to demonstrate the existence of reflections due to the internal structure of the cochlea. These are always present, but of course are most clearly seen in the absence of additional external reflections which depend on the choice of the boundary condition.

We tested a variety of termination values which included open end and short cut. Among others we used 300 μN s mm^{-3} as given in Matthews (1983) for a realistic boundary value at the stapes and also 150 μN s mm^{-3}, which is the specific cochlea input impedance (de Boer, 1980). All these terminations will give rise to external reflections. For the calculations presented here we used the value of the classical characteristic impedance $C(x,\omega)$, as defined in Eq. (6) below, at the *stapes* and *helicotrema*, respectively, as starting values. This is done to avoid additional disturbances due to boundary effects, as this choice yields the lowest external reflections.

On an infinite, homogeneous transmission line the characteristic impedance $C(x,\omega)$ and local impedances $\mathscr{R}(x,\omega)$ and $\mathscr{L}(x,\omega)$ are equal and constant for arbitrary values of x. On a finite, homogeneous transmission line terminated with its characteristic impedance $C(x,\omega)$, the local voltage and current conditions, i.e., the local impedances, are unchanged compared to the infinite line. Only this type of termination is free of reflections (Meinke, 1968).

At locations where the line's parameters are varying slowly, the concept of a "characteristic impedance" may still be valid. For reflection-free transmission at these locations we will therefore expect the line's "characteristic impedance" to be again in fairly good agreement with local impedances. The "characteristic impedance," evaluated from the local values of the parameters Z and Y, should then approximate $\mathscr{R}(x,\omega)$ as well as $\mathscr{L}(x,\omega)$. Therefore, we should expect the latter two to be nearly equal to one another. We will call the so computed function the "classical characteristic impedance,"

$$C(x,\omega) = \sqrt{Z(x,\omega)/Y(x,\omega)}. \qquad (6)$$

Moreover, $\mathscr{R}(x,\omega)$ should be well approximated by the "improved classical impedance,"

$$Q(x,\omega) = \frac{\sqrt{Z(x,\omega)/Y(x,\omega)}}{1 - i/2[k'(x,\omega)/k^2(x,\omega)]} \qquad (7)$$

with

$$k(x,\omega) = \sqrt{Z(x,\omega)Y(x,\omega)},$$

as derived in de Boer and Viergever (1986) on the basis of the Liouville–Green approximation.

The numerical results discussed below were obtained using a fourth-order Runge–Kutta algorithm for first-order differential equations. Convergence tests were made for up to 300 mesh points per mm. Solutions were checked to be stable for variations in the boundary conditions. Ten mesh points per mm were found to be sufficiently accurate. This is the number of points used in our computations. With this setup, computer time amounts to about 0.1 s on a VAX 11/750.

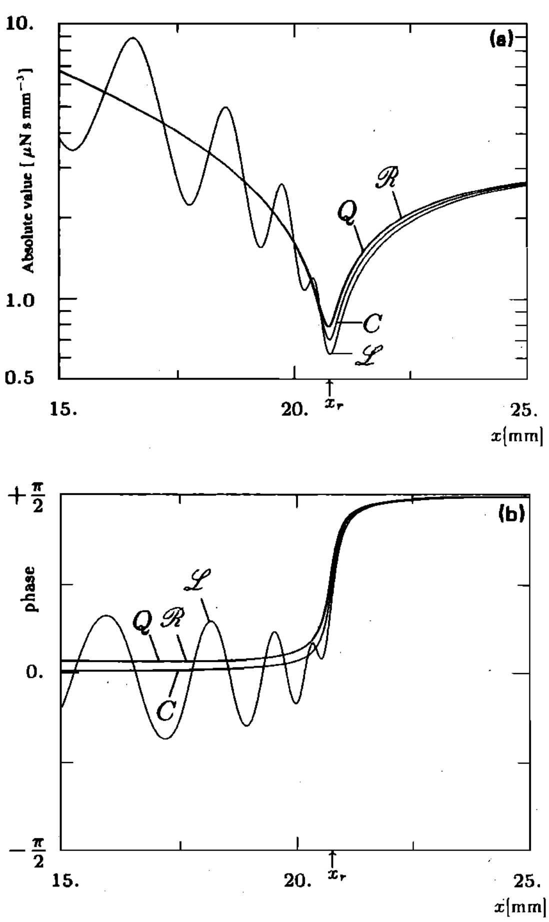

FIG. 2. Absolute value (a) and phase (b) of the two "local" impedances $\mathscr{R}(x,\omega)$ (i.e., the one measured to the right) and $\mathscr{L}(x,\omega)$ (i.e., the one measured to the left) as compared to the "classical" impedance $C(x,\omega)$ and the "improved classical" impedance $Q(x,\omega)$. The angular frequency ω of the excitation is 2π kHz. The resonance point, i.e., the location where the basilar membrane impedance is real, is $x_r = 20.76$ mm. Note that $\mathscr{R}(x,\omega)$ and $Q(x,\omega)$ cannot be distinguished on this scale.

III. NUMERICAL RESULTS

Figure 2 presents the numerical solutions for $\mathscr{R}(x,\omega)$ and $\mathscr{L}(x,\omega)$ as compared to $C(x,\omega)$ and $Q(x,\omega)$ as functions of x for fixed angular frequency $\omega = 2\pi$ kHz of the excitation. A representation as functions of the frequency for a fixed location would—with appropriate scaling—show the same picture. Figure 2(a) depicts the absolute values and Fig. 2(b) depicts the phases.

The values of $Q(x,\omega)$ and $\mathscr{R}(x,\omega)$ are seen to be indistinguishable on the scale chosen. This means that the Liouville–Green approximation is sufficiently accurate to predict the $\mathscr{R}(x,\omega)$-type of local impedance. The above-mentioned equality of $\mathscr{R}(x,\omega)$, $\mathscr{L}(x,\omega)$, and $C(x,\omega)$, however, holds only in the direct vicinity of the resonance point x_r and in the region to the right of it.

In the region left of the resonance point, $\mathscr{L}(x,\omega)$ shows wide deviations from $C(x,\omega)$, leaving only $\mathscr{R}(x,\omega)$ in agreement with the line's "characteristic impedance." These deviations betray that the left-going wave undergoes considerable reflection on its path beyond the resonance point, whereas right-traveling waves remain free of reflections. This "internal" reflection is not due to the termination but due to the structure of the cochlea. It is thus inevitable that a spontaneous emitter of sound is submitted to the reflections of its own signal, even if the termination of the cochlea is producing little external reflection. This feedback is due to waves traveling backward towards the *stapes* and being reflected internally by the inhomogeneity of the cochlea. This will influence the stability of such an emitter. In any event, the approximation by $Q(x,\omega)$ is no longer valid here: The reflections of the left-going wave are too severe to allow the Liouville–Green method to be applicable.

ACKNOWLEDGMENTS

The authors wish to express their gratitude to Professor E. de Boer (Amsterdam, The Netherlands) and Professor V. Rittenberg (Bonn, Federal Republic of Germany), thanks to whom the text was improved markedly. We thank the computing center of the Academic Medical Center (Amsterdam) and Physics Institute (Bonn) for their support.

[1]By the physical properties of the parameters involved, the canonical analog of particle velocity v is given by the current density j, whereas current I corresponds to volume velocity $v \cdot A$. Here, A represents the cochlea channel's cross section, which is constant for the one-dimensional model. This makes the two analogies principally equivalent, the only difference being the involvement of an additional factor A if volume velocity is used. This will lead only to a scale transformation of Fig. 2 and does not affect the results reported in this paper. The reader should be aware that this reflects just the difference between acoustical and specific acoustical impedance.

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